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Axially gauge-covariant electrodynamics

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Abstract. A non-perturbative gauge-covariant technique, which successfully determines the infrared behaviour of propagators in electrodynamics in a class of covariant gauges, is used to obtain the infrared behaviour in axial gauges.

1. Introduction

The gauge technique (Salam 1963) is proving to be a significant tool for uncovering certain non-perturbative properties of gauge theories. In essence, the technique relies upon solving the gauge identities (Delbourgo 1978) to determine a substantial component of the multi-meson Green functions in terms of the pure-source Green functions, and then to insert the 'solutions' into the coupled Green-function equations to obtain initial values of the source amplitudes and successive transverse component improvements which are not determined by the gauge identities themselves. In electrodynamics (Delbourgo and West 1977a, Delbourgo 1977) the scheme has been shown to provide a very efficient short cut to the infrared (and initial ultraviolet) behaviour (Delbourgo and West 1977b) of the charged lines in the class of covariant gauges specified by the gauge-fixing term $(\partial A)^2/2a$. However, in so far as relativistically covariant gauges make for complicated identities when the internal symmetry group is non-Abelian, the gauge technique is not easily adapted to obtain the infrared behaviour in Yang–Mills theory, for instance. It is preferable in that case to adhere to axial gauges where the identities involve no fictitious terms (Delbourgo *et al* 1974, Kummer 1975) and are capable of solution. The purpose of this paper is to determine the full infrared behaviour in axial gauge electrodynamics by means of the gauge technique as a preliminary investigation to the more difficult non-abelian problem.

The axial gauge is specified by the constraint $n \,.\, A = 0$, where n is an arbitrary four-vector. (Since the normalisation of n is irrelevant to the on-shell matrix elements, which are n-independent and relativistically covariant, one can choose n to be time-like and of unit length, $n^2 = 1$, as we subsequently do.) The bare photon propagator in general reads

$$D_{\mu\nu}(k) = \left(-\eta_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n \,.\, k} - \frac{n^2 k_\mu k_\nu}{(n \,.\, k)^2}\right)\frac{1}{k^2 + \mathrm{i}\epsilon} \tag{1}$$

where a principal-part prescription must be taken to avoid unphysical singularities from the vanishing of $n \,.\, k$. Note that $n^\mu D_{\mu\nu} = 0$ even for the complete propagator. The resulting quantum corrections then lead to Green functions which depend on the momentum invariants $p \, . \, q$, etc, as well as their n-projections $p \, . \, n$, $q \, . \, n$, etc, when one goes off mass-shell. Thus off-shell amplitudes are not fully relativistically covariant but only enjoy a limited O(3) covariance, connected with the little group of n†. This makes computation perhaps more involved (Frenkel and Taylor 1976) than would be the case for Lorentz covariant gauges, but on the other hand the gauge identities, whether the symmetry group of the Lagrangian is abelian or not, assume their naive canonical form and this nice feature facilitates the application of the gauge technique. The method of solution of the identities is based upon a spectral weighting of the classical theory and has been detailed in an earlier paper (Delbourgo 1978); here we shall merely quote the relevant formulae as they apply to electrodynamics. Let Δ and S stand for the scalar and spinor propagators. When $(p \, . \, n)^2 < m^2$ (threshold) there are the spectral representations‡

$$\Delta(p) = \Delta(p^2, p \, . \, n) = \int \frac{\rho(W^2, p \, . \, n) \, W^2}{p^2 - W^2 + \mathrm{i}0} \tag{2a}$$

$$S(p) = S(\gamma \, . \, p, \gamma \, . \, n) = \int \frac{\rho(W, p \, . \, n) + \gamma \, . \, n \, \rho_0(W^2, p \, . \, n)}{\gamma \, . \, p - W + \mathrm{i}0\epsilon(W)} \, \mathrm{d}W \tag{2b}$$

and the gauge technique then supplies gratuitously the starting vertex functions

$$\begin{aligned} &\Delta(p)\Gamma_\mu(p, p') \, \Delta(p') \\ &\quad = \tfrac{1}{2} \int \mathrm{d}W^2 \frac{1}{p^2 - W^2} (p + p')_\mu \frac{1}{p'^2 - W^2} (\rho(W^2, p \, . \, n) + \rho(W^2, p' \, . \, n)) \\ &\quad - \tfrac{1}{2} \int \mathrm{d}W^2 n_\mu \left(\frac{1}{p^2 - W^2} + \frac{1}{p'^2 - W^2} \right) \frac{\rho(W^2, p \, . \, n) - \rho(W^2, p' \, . \, n)}{p \, . \, n - p' \, . \, n} \end{aligned} \tag{3a}$$

and

$$\begin{aligned} &S(p)\Gamma_\mu(p, p')S(p') \\ &\quad = \tfrac{1}{2} \int \mathrm{d}W \frac{1}{\gamma \, . \, p - W} \gamma_\mu \frac{1}{\gamma \, . \, p' - W} (\rho(W, p \, . \, n) + \gamma \, . \, n\rho_0(W^2, p \, . \, n) \\ &\quad + \rho(W, p' \, . \, n) + \gamma \, . \, n\rho_0(W^2, p' \, . \, n)) - \tfrac{1}{2} \int \mathrm{d}W n_\mu \left(\frac{1}{\gamma \, . \, p - W} + \frac{1}{\gamma \, . \, p' - W} \right) \\ &\quad \times \frac{\rho(W, p \, . \, n) + \gamma \, . \, n\rho_0(W^2, p \, . \, n) - \rho(W, p' \, . \, n) - \gamma \, . \, o\rho_0(W^2, p' \, . \, n)}{p \, . \, n - p' \, . \, n} \end{aligned} \tag{3b}$$

Similar formulae can be worked out as additional photons are emitted. The expressions (3) can be substituted directly into the Dyson–Schwinger equations to obtain integral equations determining the spectral functions ρ. The salient points involved in computing these ρ are exposed in the next two sections. Here we shall merely quote the

† If one chooses n to be space-like, the functions become O(2, 1) covariant instead, while if n is light-like, E(2) becomes the relevant group.

‡ When the Lagrangian is formally invariant under charge conjugation, with $n \to -n$, we obtain $S(p, n) = C^{-1}\tilde{S}(-p, -n)C$ for the two-point function, which eliminates the covariant $[\gamma \, . \, p, \gamma \, . \, n]$. In the representation (2b) the order of $\gamma \, . \, n$ and $\gamma \, . \, p$ is irrelevant because ρ_0 is taken even in W. Such representations were proved by Johnson (1960) in the radiation gauge, and since the proofs rely solely on rotation invariance they can be extended to $A_0 = 0$. Hagen (1963) has used them extensively.

non-perturbative answers for the ensuing propagators in the infrared limit:

$$\Delta(p) \sim (p^2-m^2)^{-1}(p^2/m^2-1)^{\xi}$$

$$S(p) \sim \tfrac{1}{2}(\gamma\,.\,p-m)^{-1}[(p^2/m^2-1)^{\xi_+}+(p^2/m^2-1)^{\xi_-}] + \gamma_0 m(p^2-m^2)^{-1}[(p^2/m^2-1)^{\xi_+}-(p^2/m^2-1)^{\xi_-}] \quad (4)$$

where

$$\xi = -\frac{e^2}{2\pi^2}\left(1-\frac{\tan^{-1} b}{b}\right) \qquad \xi_\pm = -\frac{e^2}{2\pi^2}\frac{m}{m \pm p\,.\,n}\left(1-\frac{\tan^{-1} b}{b}\right)$$

$$b = [m^2/(p\,.\,n)^2-1]^{1/2}.$$

It is interesting that near the mass shell all non-covariant behaviour gets absorbed in the exponents; this represents the main result of this paper.

The equations for ρ seem quite intractable away from $p^2 = m^2$ and we have as yet been unable to discover the complete solution for all p^2. Nevertheless, in that the infrared structure of Yang–Mills theory is believed to hold the key of the confinement mechanism for chromodynamics, we feel that an extension of our calculation to the non-Abelian problem ought to be possible and might shed some light on that question. We will leave that as a separate investigation as we suspect the problem is an order of magnitude more difficult.

2. Scalar electrodynamics

It is worth going through this case in a little detail to bring out the difficulties connected with the non-relativistic covariance. The same problems are inherent in the spinor case, but as well there occur γ-matrix complications causing tricky coupled integral equations. Such problems are absent for scalars. For ease of calculation we shall direct n along the time axis so that the Green functions possess spatial rotation invariance only, being functions of energies and three-vector products; it is easy enough to reinstate the n dependence of the amplitudes in the end for more arbitrary n directions by replacing p_0^2 with $(p\,.\,n)^2/n^2$, etc. When $n = (1, 0)$ the only non-vanishing components of the photon propagator are

$$D_{ij}(k) = (\delta_{ij} - k_i k_j/k_0^2)/k^2$$

and the complete scalar propagator $\Delta(p)$ becomes a function of p^2 and p_0^2. Also in (3*a*) we may then neglect the time component of Γ_μ to all intents and purposes. This is because Γ_μ is contracted against a polarisation vector or else multiplies a propagator which is also orthogonal to n. Thus the Dyson–Schwinger equation† for the charged scalar line reads

$$Z^{-1} = (p^2-m_0^2)\Delta(p) - \tfrac{1}{2}\mathrm{i}\,e^2\int \mathrm{d}W^2\,\bar{\mathrm{d}}^4k\,(\rho(W^2,p_0)+\rho(W^2,p_0-k_0)) \times\frac{(2p-k)_i(\delta_{ij}-k_ik_j/k_0^2)(2p-k)_j}{k^2(p^2-W^2)[(p-k)^2-W^2]}$$

† In this context it is essential to note that the mass-shell element δm^2 or $\Pi(m^2, p_0, m^2)$ is in fact independent of p_0.

which may be rewritten in the form

$$Z^{-1}=\int \frac{dW^2}{p^2-W^2}\Big((p^2-m_0{}^2)\rho(W^2,p_0)$$
$$+\tfrac{1}{2}\int dk_0\,\Pi_{k0}(p^2,p_0,W^2)(\rho(W^2,p_0)+\rho(W^2,p_0-k_0))\Big). \tag{5}$$

In (5) we have represented the lowest-order self-energy correction appropriate to a scalar of mass W in the axial gauge by

$$\Pi(p^2,p_0,W^2)=\int dk_0\,\Pi_{k0}(p^2,p_0,W^2).$$

The difference between (5) and the corresponding equation for a covariant gauge choice lies in the occurrence of the spectral function ρ for the intermediate scalar line, which prevents the integration over the intermediate energy being carried out immediately. If we take the imaginary part of (5) carrying out our renormalizations so that finally†

$$\Pi(p^2,p_0,m^2)\rightarrow -\frac{(p^2-m^2)}{\pi}\int \frac{\operatorname{Im}\Pi(p^2,p_0,W^2)\,dW^2}{(p^2-W^2)(W^2-m^2)} \tag{6}$$

the spectral function equation reduces to

$$(W^2-m^2)\rho(W^2,p_0)=\frac{1}{2\pi}\int dW'^2\,dk_0(\rho(W'^2,p_0)$$
$$+\rho(W'^2,p_0-k_0))\frac{\operatorname{Im}\Pi_{k0}(W^2,p_0,W'^2)}{(W^2-W'^2)}. \tag{7}$$

At this stage some comments about the self-energy Π have to be made. The singularities of the integral

$$\int d^4k(k^2+i0)^{-1}[(p-k)^2-m^2+i0]^{-1}\mathscr{P}(k_0^{-2})$$

in the k_0 plane become confluent at $k_0=|\boldsymbol{k}|=p_0-[(\boldsymbol{p}-\boldsymbol{k})^2+m^2]^{1/2}$, which amounts to a vanishing of k^2 and $[(p-k)^2-m^2]$ along the future light cones. This covariant pinch leads to

$$\operatorname{Im}\Pi(p)=\frac{e^2}{2\pi^2}\int d^4k\,\delta_+(k^2)\delta_+[(p-k)^2-m^2](\boldsymbol{p}^2-(\boldsymbol{p}\,.\,\boldsymbol{k})^2/\boldsymbol{k}^2)$$
$$=\int_{k_{0-}}^{k_{0+}} dk_0\operatorname{Im}\Pi_{k0}(p^2,p_0,m^2). \tag{8a}$$

A second type of confluence can occur at $k_0=0=p_0\pm[(\boldsymbol{p}-\boldsymbol{k})^2+m^2]^{1/2}$ and is visibly non-covariant. It can be avoided if the integral (8a), which is a function of p^2 and p_0, is continued to $p_0^2<m^2$, and this means that we are ultimately evaluating (8a) for imaginary values of $|\boldsymbol{p}|$ in order to avoid non-covariant discontinuities. The final results can be continued to real values of $|\boldsymbol{p}|$ if any singularities are correctly circumvented

† In this context it is essential to note that the mass-shell element δm^2 or $\Pi(m^2,p_0,m^2)$ is in fact independent of p_0.

by the iϵ prescriptions. In (8a) the integration limits are

$$k_{0\pm}=(p^2-m^2)/2(p_0\mp|p|) \tag{8b}$$

and

$$\operatorname{Im}\Pi_{k_0}(p^2,p_0,m^2)=\frac{e^2|p|}{4\pi}\left[1-\left(\frac{2p_0k_0-p^2+m^2}{2|p|k_0}\right)^2\right]\theta(p^2-m^2). \tag{8c}$$

Upon changing variable in (7) from k_0 to u, by means of the substitution

$$k_0=(W^2-W'^2)/2(p_0-|\boldsymbol{p}|u)$$

the integral equation for the spectral function ρ simplifies to

$$(W^2-m^2)\rho(W^2,p_0)=\frac{e^2}{16\pi^2}\int_{m^2}^{W^2}\mathrm{d}W'^2\int_{-1}^{1}\mathrm{d}u\,\frac{(1-u^2)}{(u-p_0/|\boldsymbol{p}|)^2}$$
$$\times\left(\rho(W'^2,p_0)+\rho\left(W'^2,p_0-\frac{(W^2-W'^2)}{2(p_0-|\boldsymbol{p}|u)}\right)\right). \tag{9}$$

It is by no means trivial to solve because of the complicated u dependence entering within the non-covariant argument of ρ. However, in the infrared limit it does become amenable because $W^2\to W'^2\to m^2$ on the right of (9). Therefore

$$(W^2-m^2)\rho(W^2,p_0)\to\frac{e^2}{4\pi^2}\int_{m^2}^{W^2}\mathrm{d}W'^2\rho(W'^2,p_0)\left(-2+\frac{p_0}{|\boldsymbol{p}|}\ln\left(\frac{p_0+|\boldsymbol{p}|}{p_0-|\boldsymbol{p}|}\right)\right). \tag{10}$$

In the same limit, $|\boldsymbol{p}|/p_0\to \mathrm{i}(m^2/p_0{}^2-1)^{1/2}\equiv \mathrm{i}b$, and we readily obtain the answer

$$\rho(W^2,b)\simeq(W^2-m^2)^{-1+(e^2/2\pi^2)(-1+b^{-1}\tan^{-1}b)} \tag{11}$$

which can be compared† with the covariant gauge result (Hagen 1963, Zwanziger 1975)

$$\rho(W^2,a)\simeq(W^2-m^2)^{-1+e^2(a-3)/8\pi^2}. \tag{12}$$

The propagator $\Delta(p)$ behaves similarly. At this level, the non-relativistic character of the axial gauge is exhibited by the nature of the exponent combination

$$(b^{-1}\tan^{-1}b-1)=\int_0^1\mathrm{d}u[1+u^2(p^2/(p.n)^2-1)]^{-1}. \tag{13}$$

Of course, we cannot expect things to remain so simple an appreciable distance away from the mass shell.

3. Spinor electrodynamics

Here we are faced with integral equations for three spectral functions in general. This can be understood by setting

$$\rho+\gamma.n\rho_0=\epsilon(W)(W\rho_1(W^2,p.n)+m\rho_2(W^2,p.n)+\gamma.n\,p.n\,\rho_3(W^2,p.n))$$

when (2b) can be rewritten in the more conventional Lehmann form

$$S(p)=\int_{m^2}^{\infty}\frac{\gamma.p\,\rho_1(W^2,p.n)+m\rho_2(W^2,p.n)+\gamma.n\,p.n\,\rho_3(W^2,p.n)}{p^2-W^2+\mathrm{i}0}\mathrm{d}W^2 \tag{14}$$

† The Yennie gauge $a=3$ finds its parallel in $p.n\to m$ for the axial gauge.

where ρ_1, ρ_2 and ρ_3 are even in W (and also $p\,.\,n$ in fact). These three ρ_i become coupled via the Dyson–Schwinger equation

$$Z^{-1}=S(p)(\gamma\,.\,p-m_0)-\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\,S(p)\Gamma_i(p,p-k)S(p-k)D^{ij}(k)\gamma_j,$$

which in gauge approximation reads

$$Z^{-1}=\int \mathrm{d}W\frac{(\rho(W,p_0)+\gamma_0\rho_0(W^2,p_0))}{\gamma\cdot p-W}(\gamma\,.\,p-m_0)$$
$$-\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^2\int \mathrm{d}W\int \bar{\mathrm{d}}^4k\frac{1}{\gamma\,.\,p-W}\gamma_i\frac{1}{\gamma\,.\,(p-k)-W}D^{ij}(k)\gamma_j$$
$$\times(\rho(W,p_0)+\gamma_0\rho_0(W^2,p_0)+\rho(W,p_0-k_0)+\gamma_0\rho_0(W^2,p_0-k_0)).\qquad(15)$$

To obtain the solutions for ρ, let $\Sigma(p,m)$ stand for the lowest-order self-energy in the axial gauge for a fermion of mass m. After mass renormalisation, we interpret

$$\Sigma(p,m)=-(\gamma\,.\,p-m)\frac{1}{\pi}\int\frac{\mathrm{Im}\,\Sigma(W',p_0,m)\,dW'}{(\gamma\,.\,p-W')(W'-m)}.$$

Now, in what follows, we shall need information about $\mathrm{Im}\,\Sigma$. For $p_0{}^2<m^2$ we have a covariant discontinuity in the self-energy†, and it is easy enough to arrive at the answer

$$\mathrm{Im}\,\Sigma=(\mathrm{e}^2/8\pi^2)\int \mathrm{d}^4k\,\delta_+(k^2)\,\delta_+[(p-k)^2-m^2]\gamma_i(\delta_{ij}-k_ik_j/\boldsymbol{k}^2)[\gamma\,.\,(p-k)-m]\gamma_j$$
$$\equiv\int_{k_{0-}}^{k_{0+}}\mathrm{d}k_0\,\mathrm{Im}\,\Sigma_{k_0}(p,p_0,m).\qquad(16a)$$

Actually we shall not require the full $\mathrm{Im}\,\Sigma_{k_0}$, but only the integrated form

$$\mathrm{Im}\,\Sigma=\frac{\mathrm{e}^2}{16\pi}\theta(p^2-m^2)(p^2-m^2)$$
$$\times\left[\frac{p_0\gamma_0(p^2+m^2)}{p4}-\frac{2m}{p^2}+\frac{2\boldsymbol{p}\,.\,\gamma}{\boldsymbol{p}^2}\left(-2+\frac{p_0}{|\boldsymbol{p}|}\ln\left(\frac{p_0+|\boldsymbol{p}|}{p_0-|\boldsymbol{p}|}\right)-\frac{(p^2+m^2)\boldsymbol{p}^2}{2p^4}\right)\right].\qquad(16b)$$

In terms of the self-energy, the renormalised form of the equation (15) can be rewritten as

$$0=\int \mathrm{d}W\frac{(W-m)}{\gamma\,.\,p-W}(\rho(W,p_0)+\gamma_0\rho_0(W^2,p_0))$$
$$+\frac{1}{2\pi}\int \mathrm{d}W\,\mathrm{d}k_0\,\mathrm{d}W'\frac{\mathrm{Im}\,\Sigma_{k_0}(W',p_0,W)}{(\gamma\,.\,p-W')(W-W')}(\rho(W',p_0)$$
$$+\gamma_0\rho_0(W'^2,p_0)+\rho(W',p_0-k_0)+\gamma_0\rho_0(W'^2,p_0-k_0)).$$

This is directly comparable with (5) after renormalisation. In fact, the imaginary-part

† The fermion self-energy graph has been computed dimensionally by Frenkel and Meuldermans (1976) and more recently by Konetschny (1977), who has also demonstrated explicitly the equality $Z_1=Z_2$, thereby confirming earlier calculations (Delbourgo *et* 1974)

analogue of (7) is the integral equation

$$(W-m)\epsilon(W)(\rho(W,p_0)+\gamma_0\rho_0(W^2,p_0))$$
$$=\frac{1}{2\pi}\int dW'\, dk_0(\rho(W',p_0)+\gamma_0\rho_0(W'^2,p_0)+\rho(W',p_0-k_0)$$
$$+\gamma_0\rho_0(W'^2,p_0-k_0))\operatorname{Im}\Sigma_{k_0}(W,p_0,W')/(W-W'). \tag{17}$$

We can change variable, as in the scalar case, from k_0 to u. It becomes a useful substitution only in the infrared limit, whereupon

$$(W-m)(\rho(W,p_0)+\gamma_0\rho_0(W^2,p_0))\to\frac{e^2}{16\pi^2}\int_m^W dW'(\rho(W',p_0)+\gamma_0\rho_0(W'^2,p_0))$$
$$\times(W+W')\left[\frac{p_0\gamma_0(W^2+W'^2)}{W^4}-\frac{2W'}{W^2}-\frac{4(W-p_0\gamma_0)}{p_0^2-W^2}\right.$$
$$\left.\times\left(-1+\frac{\tan^{-1}b}{b}-\frac{(W^2+W'^2)(p_0^2-W^2)}{4W^4}\right)\right]$$
$$\simeq\frac{e^2}{2\pi^2}\frac{m}{(m+\gamma_0p_0)}\left(-1+\frac{\tan^{-1}b}{b}\right)\int_m^W dW'(\rho(W',p_0)+\gamma_0\rho_0(W'^2,p_0)) \tag{18}$$

using the old notation $b^2=p^2/p_0^2-1$. Equation (18) is easy to solve. One merely introduces projections

$$\rho+\gamma_0\rho_0=\tfrac{1}{2}(1+\gamma_0)\rho_++\tfrac{1}{2}(1-\gamma_0)\rho_-$$

to get the uncoupled differential equations

$$\frac{d}{dW}[(W-m)\rho_\pm]\simeq\frac{e^2}{2\pi^2}\frac{m}{m\pm p_0}\left(-1+\frac{\tan^{-1}b}{b}\right)\rho_\pm\equiv\xi_\pm(p_0/m)\rho_\pm$$

whence

$$\rho_\pm\propto R_\pm(W-m)^{-1+\xi_\pm(p_0/m)}. \tag{19}$$

The arbitrary overall constants $R_\pm$ have to be fixed equal to give the free propagator as $e^2\to0$. To this leading order as $p^2\to m^2$ we thereby obtain

$$S(p)\to\frac{(\gamma\,.\,p+m)}{2m^2}R\{[(p^2/m^2)-1]^{-1+\xi_+(b)}+[(p^2/m^2)-1]^{-1+\xi_-(b)}\}$$
$$+\frac{\gamma_0R}{m}\{[(p^2/m^2)-1]^{-1+\xi_+(b)}-[(p^2/m^2-1]^{-1+\xi_-(b)}\}. \tag{20}$$

This has been rewritten rather more neatly as equation (4) in the introduction. All we know about the normalisation factor is that $R(p_0,e)=1$ when $e=\xi_\pm=0$, to be able to recover the undressed propagator as the interaction vanishes. If one expands (20) in powers of $\xi_\pm$, one recovers the perturbative answer for S, as can be easily checked. Observe too the close similarity between (19) and (11).

Without having studied the counterpart equations for quantum chromodynamics, it is difficult to hazard a guess about the behaviour of the gluon (and quark) propagator in the infrared. However, it would be most disappointing if one found that the gauge technique merely served to exponentiate the lowest-order perturbation answer without supplying additional $1/q^2$ dependence (Pagels 1977), for it would mean that the longitudinal components of the vertex function—which are *not* determined by the gauge identities—are responsible for the conjectured confinement.

Appendix

To understand how the renormalisations are carried out in § 2, as well as in § 3, consider the simpler situation in which a covariant gauge is chosen. Then the complete scalar propagator equation is

$$Z^{-1}=(p^2-m_0{}^2)\,\Delta(p)-\mathrm{i}\,e^2\int \mathrm{d}W^2\,\bar{d}^4k\frac{\rho(W^2)}{p^2-W^2}\,\frac{(2p-k)_\mu D^{\mu\nu}(k)(2p-k)_\nu}{[(p-k)^2-W^2]}$$

$$-\mathrm{i}e^2\Delta(p)\int \bar{d}^4k\Gamma_\mu{}^T(p,p-k)D^{\mu\nu}(k)\Delta(p-k)(2p-k)_\nu$$

$$+\text{photon tadpole term}$$

$$+2e^4\int \Gamma_{\mu\nu}(p,-k;p;k')D^{\mu\lambda}(k)D^{\nu\lambda}(k')\,\bar{\mathrm{d}}^4k'\,\bar{\mathrm{d}}^4p'\,\Delta(p')\,\Delta(p) \qquad (\mathrm{A}.1)$$

where $k^\mu\Gamma_\mu{}^\mathrm{T}\equiv 0$. Thus $\Gamma_\mu{}^\mathrm{T}$ is the transverse part of Γ not contained in $(3a)$ and undetermined by the gauge identities; it is, however, related to the other Green functions through the coupled integral equations of the theory. Note that $\Gamma_\mu{}^T$ multiplies a term of order e^2 and is itself at least of order e^2 since it represents a quantum correction. As we shall be concerned with the absorptive part in what follows, the tadpole graph can be forgotten—it vanishes anyhow in the context of dimensional continuation.

It is known how to renormalise (A.1) order-by-order in perturbation theory. Hence as an equation for the renormalised absorptive part ρ, the dependence on the cut-off must disappear in the end if only renormalised Green functions are involved. Neglecting the 2γ term $2e^4\int\Gamma_{\mu\nu}\ldots$, which is softer in the infrared than the rest† (and which does not arise in the spinor electrodynamics anyway), the absorptive part of (A.1) reads

$$(p^2-m_0{}^2)\rho(p^2)=\frac{1}{\pi}\,\mathrm{Im}\int \mathrm{d}W^2\,\frac{\rho(W^2)}{(p^2-W^2+\mathrm{i}0)}\,\Pi(p^2,\,W^2)+\frac{1}{\pi}\,\mathrm{Im}(\Delta(p)\Pi^\mathrm{T}(p^2)) \qquad (\mathrm{A}.2)$$

As in the text $\Pi(p^2,\,W^2)$ refers to the lowest-order (e^2) self-energy for a meson mass W, but Π^T is the self-energy contribution arising from transverse $\Gamma_\mu{}^\mathrm{T}$ and is of order e^4 at least. Now the gauge-independent self-mass

$$\delta m^2=m_0{}^2-m^2=-\mathrm{Re}\,\Pi(m^2,\,m^2)+O(e^4).$$

Therefore (A.2) reduces to

$$(p^2-m^2+\mathrm{Re}\,\Pi(p^2,p^2)-\mathrm{Re}\,\Pi(m^2,\,m^2))\rho(p^2)$$

$$=\frac{1}{\pi}\int \mathrm{d}W^2\,\rho(W^2)\left(\frac{\mathrm{Im}\,\Pi(p^2,\,W^2)}{p^2-W^2}\right)+\frac{1}{\pi}\,\mathrm{Im}(\Delta(p)\Pi^T(p^2)). \qquad (\mathrm{A}.3)$$

† It can be taken into account if necessary as in Delbourgo (1977).

The assumption that

$$\Pi(p^2, m^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\,\Pi(W^2, m^2)\,\mathrm{d}W^2}{(W^2 - p^2 - \mathrm{i}0)}$$

and the definition that

$$\Pi(p^2, W^2) = -\mathrm{i}e^2 \int \bar{\mathrm{d}}^4 k (2p-k)^\mu D_{\mu\nu}(k)(2p-k)^\nu/[(p-k)^2 - W^2]$$

imply that there is a logarithmic infinity in $(\Pi(p^2, p^2) - \Pi(m^2, m^2)]$ of the form

$$e^2(p^2 \ln(\Lambda^2/p^2 - m^2 \ln(\Lambda^2/m^2))$$

causing a term, involving a cut-off Λ^2, of order e^4 to appear on the left-hand side of (A.3). Because such cut-off dependence does not appear anywhere on the right, *except* in $\mathrm{Im}\,(\Delta(p)\Pi^T(p^2))$, it must necessarily cancel against it, for the remainder of the equation involves finite renormalised quantities. (Observe that Π^{T} is just of the right order in e^2 to accomplish this, showing that this phenomenon cannot be accidental.) Hence the renormalised integral equation for ρ simplifies to

$$(p^2 - m^2)\rho(p^2) = \int \mathrm{d}W^2\, \rho(W^2) \frac{\mathrm{Im}\,\Pi(p^2, W^2)}{p^2 - W^2} + \text{finite contributions coming from } \Gamma^{\mathrm{T}}.$$

In the initial gauge approximation these transverse components are discarded altogether, and in the end the integral equation for the initial ρ is obtained from (A.2) by replacement of ${m_0}^2$ with m^2 and reinterpretation of $\Pi(p^2, m^2)$ as the once subtracted

$$\frac{(p - m^2)}{\pi}\int \frac{\mathrm{Im}\,\Pi(W^2, m^2)\,\mathrm{d}W^2}{(W^2 - m^2)(W^2 - p^2)}.$$

The manipulations which lead from (5) to (7) for the axial gauge are almost the same apart from the fact that p_0, which acts as the gauge parameter, makes an explicit appearance everywhere and sometimes prevents certain of the energy integrals from being carried out.

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